Preface

Introduction

This text is based on a course of thirty three lectures in geometric mechanics, taught annually by the author to fourth year undergraduates in their last term in applied mathematics at Imperial College London. The text mimics the lectures, which attempt to provide an air of immediacy and flexibility in which students may achieve insight and proficiency in using one of the fundamental approaches for solving a variety of problems in geometric mechanics. This is the Euler-Poincaré approach, which uses Lie group invariance of Hamiltonian's principle to produce symmetry-reduced motion equations and reveal their geometrical meaning. It has been taught to students with various academic backgrounds from mathematics, physics and engineering.

Each chapter of the text is presented as a line of inquiry, often by asking sequences of related questions such as, What is angular velocity? What is kinetic energy? What is angular momentum? and so forth. In adopting such an inquiry-based approach, one focuses on a sequence of exemplary problems, each of whose solutions facilitates taking the next step. The present text takes those steps, forgoing any attempt at mathematical rigour. Readers more interested in a rigorous approach are invited to consult some of the many citations in the bibliography which treat the subject in that style.

Prerequisites

The prerequisites are standard for an advanced undergraduate student. Namely, the student should be familiar with linear algebra of vectors and matrices, ordinary differential equations, multivariable calculus and have some familiarity with variational principles and canonical Poisson brackets in classical mechanics at the level of a second or third year undergraduate in mathematics, physics, and engineering. An undergraduate background in physics is particularly helpful, because all the examples of rotating, spinning and rolling rigid bodies treated here from a geometric viewpoint are familiar from undergraduate physics classes.

How to read this book

Most of the book is meant to be read in sequential order from front to back. The 120 Exercises and 55 Worked Answers are indented and marked with ★ and ▲, respectively.

Key theorems, results and remarks are placed into frames (like this one).

The three appendices provide supplementary material, such as condensed summaries of the essentials of manifolds (Appendix A) and Lie groups (Appendix B) for students who may wish to acquire a bit more mathematical background. In addition, the appendices provide material for supplementary lectures that extend the course material. Examples include variants of rotating motion that depend on more than one time variable, as well as rotations in complex space and in higher dimensions in Appendix C. The appendices also contain ideas for additional homework and exam problems that go beyond the many exercises and examples sprinkled throughout the text.

Description of contents

Galilean relativity and the idea of a uniformly moving reference frame are explained in Chapter 1. Rotating motion is then treated in

Chapters 2, 3 and 4, first by reviewing Newton's and Lagrange's approaches, then following Hamilton's approach via quaternions and Cayley-Klein parameters, not Euler angles.

Hamilton's rules for multiplication of quaternions introduced the adjoint and coadjoint actions that lie at the heart of geometric mechanics. For the rotations and translations in $\mathbb{R}^3$ studied in Chapters 5 and 6, the adjoint and coadjoint actions are both equivalent to the vector cross product. Poincaré [Po1901] opened the field of geometric mechanics by noticing that these actions define the motion generated by any Lie group.

When applied to Hamilton's principle defined on the tangent space of an arbitrary Lie group, the adjoint and coadjoint actions studied in Chapter 6 result in the Euler-Poincaré equations derived in Chapter 7. Legendre transforming the Lagrangian in Hamilton's principle summons the Lie-Poisson Hamiltonian formulation of dynamics on a Lie group. The Euler-Poincaré equations provide the framework for all of the applications treated in this text. These applications include finite dimensional dynamics of three-dimensional rotations and translations in the special Euclidean group $SE(3)$. The Euler-Poincaré problem on $SE(3)$ recovers Kirchhoff's classic treatment in modern form of the dynamics of an ellipsoidal body moving in an incompressible fluid flow without vorticity.

The Euler-Poincaré formulation of Kirchhoff's problem on $SE(3)$ in Chapter 7 couples rotations and translations, but it does not yet introduce potential energy. The semidirect-product structure of $SE(3)$, however, introduces the key idea for incorporating potential energy. Namely, the same semidirect-product structure is also invoked in passing from rotations of a free rigid body to rotations of a heavy top with a *fixed* point of support under gravity. Thereby, semidirect-production reduction becomes a central focus of the text.

The heavy top treated in Chapter 8 is a key example, because it introduces the dual representation of the action of a Lie algebra on a vector space. This is the diamond operation $(\diamond)$, by which the forces and torques produced by potential energy gradients are represented in the Euler-Poincaré framework in Chapters 9, 10 and 11. The diamond operation $(\diamond)$ is then found in Chapter 12 to lie at the heart of the standard (cotangent-lift) momentum map.

This observation reveals the relation between the results of reduction by Lie symmetry on the Lagrangian and Hamiltonian sides. Namely,

> Lie symmetry reduction on the Lagrangian side produces the Euler-Poincaré equation, whose formulation on the Hamiltonian side as a Lie-Poisson equation governs the dynamics of the momentum map associated with the cotangent lift of the Lie-algebra action of that Lie symmetry on the configuration manifold.

The primary purpose of this book is to explain that statement, so that it may be understood by undergraduate students in mathematics, physics and engineering.

In the Euler-Poincaré framework, the adjoint and coadjoint actions combine with the diamond operation to provide a powerful tool for investigating other applications of geometric mechanics, including nonholonomic constraints discussed in Chapter 13. In the same chapter, nonholonomic mechanics is discussed in the context of two classic problems, known as Chaplygin's top (a rolling ball whose mass distribution is not symmetric) and Euler's disk (a spinning, falling, rolling, flat coin). In these classic examples, the semidirect-product structure couples rotations, translations and potential energy together with the rolling constraint.

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